Obtaining GHZ states from graph states

To extract or not to extract

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$$\mathcal{P}_n = \{1, -1, +i, -i\} \otimes \{I, X, Y, Z\}^{\otimes n}$$

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$$I \rightarrow \begin{bmatrix} 0 \\ \hline 0 \end{bmatrix}$$

$$X \rightarrow \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix}$$

$$Y \rightarrow \begin{bmatrix} 1 \\ \hline 1 \end{bmatrix}$$

$$Z \rightarrow \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix}$$

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$$|\psi\rangle\langle\psi| = \sum_{\sigma \in \mathcal{S}} \sigma$$

Generator matrix

$$\{g_1, g_2, \ldots, g_n\} \to \{\overrightarrow{g_1}, \overrightarrow{g_2} \ldots \overrightarrow{g_n}\} \to G = \left[\begin{array}{c} \mathcal{Z} \\ \hline \mathcal{X} \end{array}\right] \in \mathbb{F}_2^{2n\times n}$$

Transformations of stabilizer states

$$|\psi\rangle \to U\,|\psi\rangle$$

$$\Sigma_{\sigma\in\mathcal{S}}\sigma \to \Sigma_{(\sigma\in\mathcal{S})}U\sigma U^\dagger$$

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$$\mathcal{C}_n = \langle CZ_{ij}, \sqrt{X}_i, \sqrt{Z}_i\rangle$$

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$$\mathcal{C}_n^l = \langle\sqrt{X}_i, \sqrt{Z}_i\rangle$$

Binary picture

$$G = \left[\frac{\mathcal{Z}}{\mathcal{X}}\right] \to G' = QG$$

$$Q^T P Q = P$$

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Binary picture

$$G = \left[\frac{\mathcal{Z}}{\mathcal{X}}\right] \rightarrow G' = QG$$

$$Q^T P Q = P$$

$$\overset{?}{\exists} C : \ \rho' = C \rho C^\dagger$$

$$\overset{?}{Q} : G' \hat{=} QG$$

Local Clifford equivalence

$$Q^L = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right], \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \text{ all diagonal}$$

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$$\left[\begin{array}{c|c} \mathcal{Z}' & \mathcal{X}' \end{array}\right] P \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] \left[\begin{array}{c} \mathcal{Z} \\ \hline \mathcal{X} \end{array}\right] = 0$$

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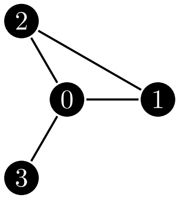
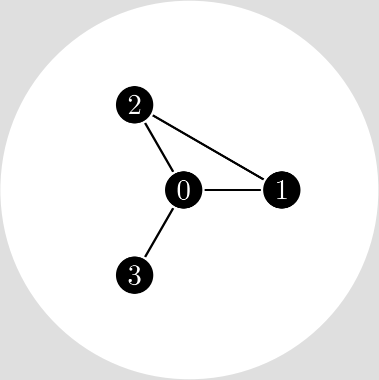
$$\left[\ \mathcal{Z}' \mid \mathcal{X}'\ \right] P \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] \left[\begin{array}{c} \mathcal{Z} \\ \hline \mathcal{X} \end{array}\right] = 0$$

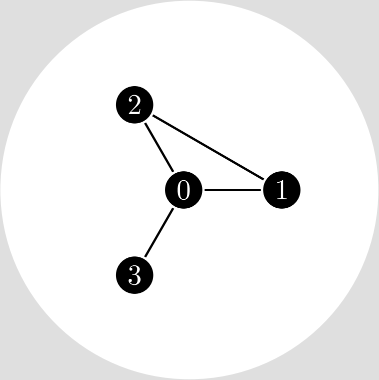
$$Q^T P Q = P$$


2
0
1
3



$G = (V, E)$

- *nodes V*
- *edges E*



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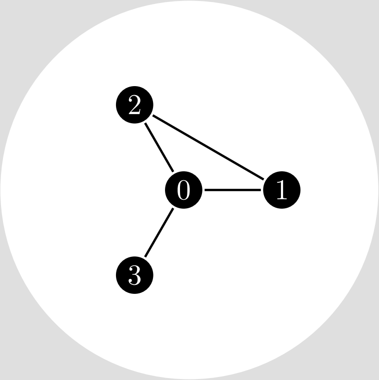
$$N_i = \{j \in V | (i, j) \in E\}$$



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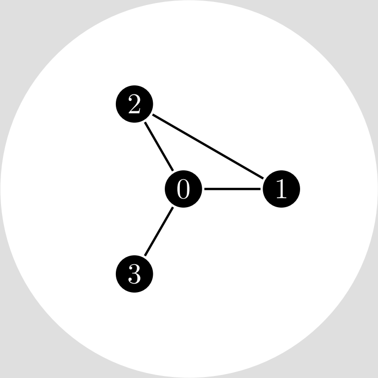
$\Theta \in \mathbb{F}_2^{|V| \times |V|} : \Theta_{ij} = 1 \leftrightarrow (i, j) \in E$



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$K[V]$ *complete* graph

Graph states

$$G = (V, E) \rightarrow |G\rangle$$

- $V \rightarrow |+\rangle^{\otimes V}$
- $(i, j) \in E \rightarrow CZ_{i,j}$

Graph states

$$|G\rangle = \prod_{(i,j)\in E} CZ_{i,j} \, |+\rangle$$

$|G\rangle$ is stabilizer

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- $\{g_i\} = \{X^i Z^{N_i}\}_i$
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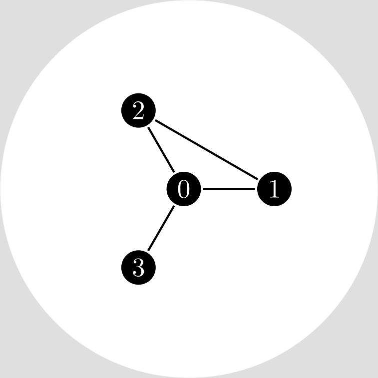
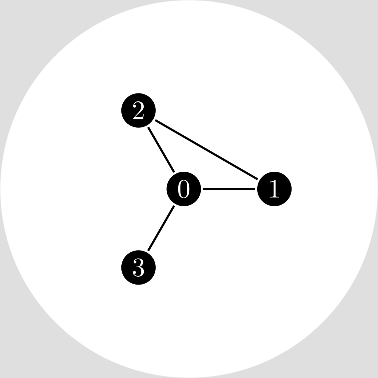
- $\{g_i\} = \{X^i Z^{N_i}\}_i$
- $G = \left[\begin{array}{c}\Theta \\ \hline I\end{array}\right]$
- Every stabiliser state is **LC-equivalent** to ‘a’ graph state

Example





$$|N\rangle = CZ_{0,1}CZ_{0,2}CZ_{0,3}CZ_{1,2}\,|+,+,+,+\rangle$$

$$\begin{aligned} g_1 &= XZZZ \\ g_2 &= ZXZI \\ g_3 &= ZZXI \\ g_4 &= ZIIX \end{aligned}$$

Linear cluster states

$$|\mathrm{L}\rangle_n = \prod_i CZ_{i,i+1} |+, \dots, +\rangle$$

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$|\mathrm{L}\rangle_N$

GHZ STATES

$$|\mathrm{GHZ}\rangle_n = |0,\ldots,0\rangle + |1,\ldots,1\rangle$$

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$$|\mathrm{GHZ}\rangle_n \hat{=} |0, +, \ldots, +\rangle + |1, -, \ldots, -\rangle$$

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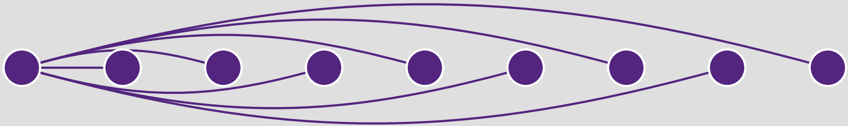
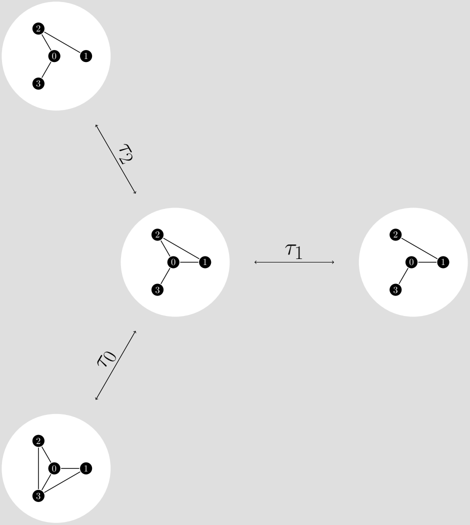
$|\mathrm{GHZ}\rangle_n$



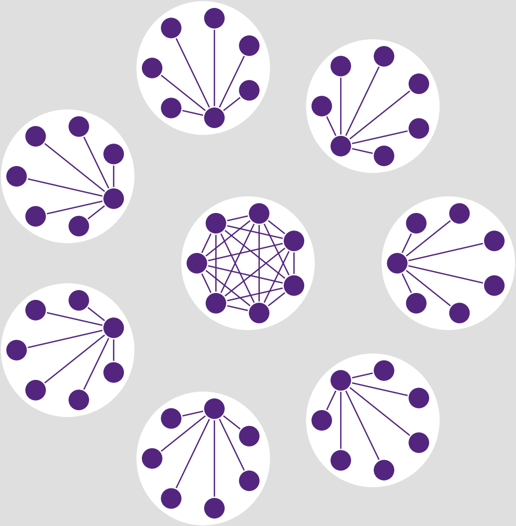
Local complementation

$$G \xrightarrow{\tau_i} G \oplus \mathcal{K}[G[N_i]]$$

Local complementation



Orbits



Why?

- Every stabilizer state is $\mathcal{C}^1$-equivalent to a graph state

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- $|G\rangle \xrightarrow{\mathcal{C}^1} |G'\rangle \leftrightarrow G' \in Orb(G)$

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- $|G\rangle \xrightarrow{\mathcal{C}^1} |G'\rangle \leftrightarrow G' \in Orb(G)$
- $\mathbf{A}\Theta + \mathbf{B} + \Theta'\mathbf{C}\Theta + \Theta'\mathbf{D} = \mathbf{0}$

Making the problem more interesting

From only $\mathcal{C}^l$ to

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- Operations from $\mathcal{C}^{\text{two}-1}$

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- Operations from $\mathcal{C}^{\mathrm{two-l}}$
- Limiting the classical communication

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- Local (Pauli) measurements

Making the problem more interesting

From only $\mathcal{C}^l$ to

- Operations from $\mathcal{C}^{\mathrm{two-l}}$
- Limiting the classical communication
- Local (Pauli) measurements $\rightarrow$ node deletions

Extracting graph states
…. from graph states

Goal

- Map $|\mathrm{L}\rangle_n$ to $|\mathrm{GHZ}\rangle_k$

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- Map $|\mathrm{L}\rangle_n$ to $|\mathrm{GHZ}\rangle_k$
- Using only local measurements and rotations

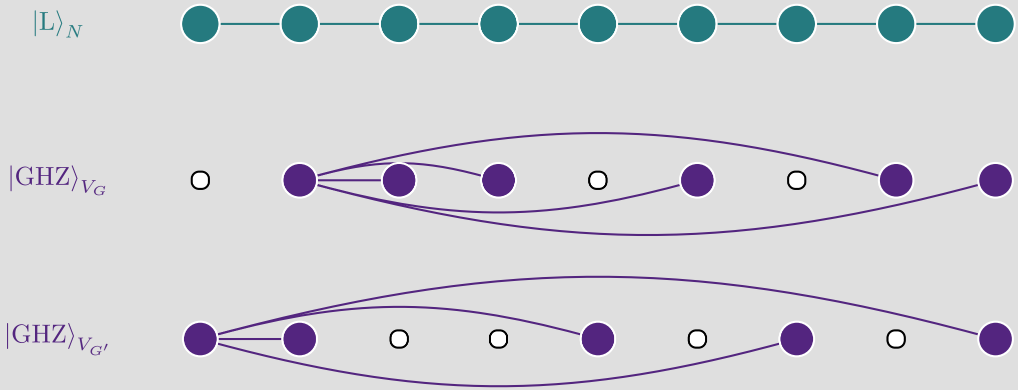
Goal

- Map $|\mathrm{L}\rangle_n$ to $|\mathrm{GHZ}\rangle_k$
- What k are possible?

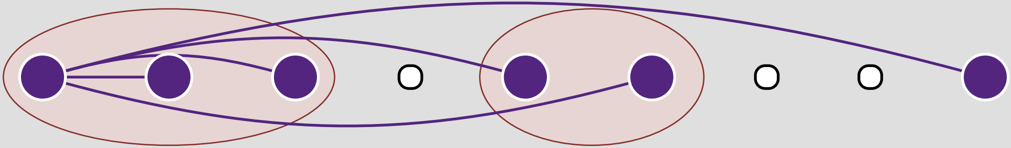
Goal

- Map $|\mathrm{L}\rangle_n$ to $|\mathrm{GHZ}\rangle_k$
- What *node selections* are possible?

Node selections?



Islands



Imbossibility results

- No islands of size ≤ 3

- 2-islands only at the *edge*

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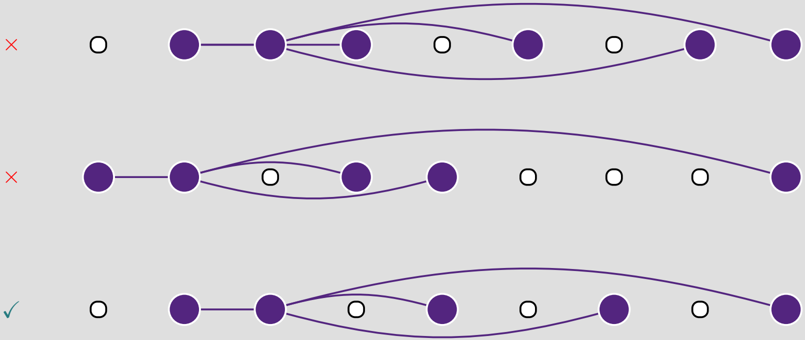
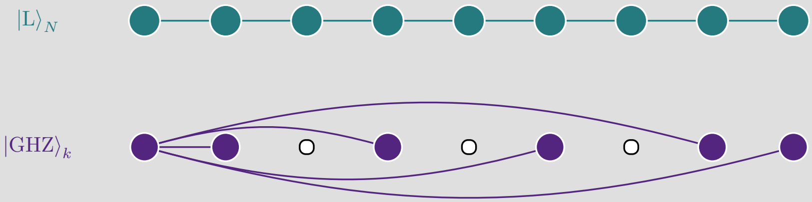
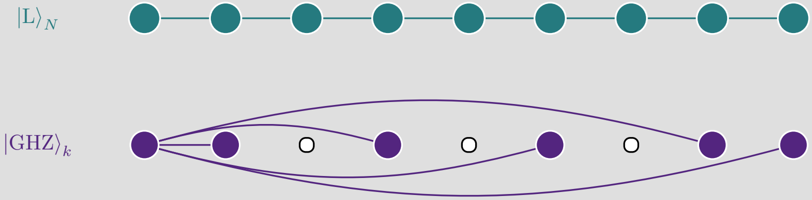


Upper bound

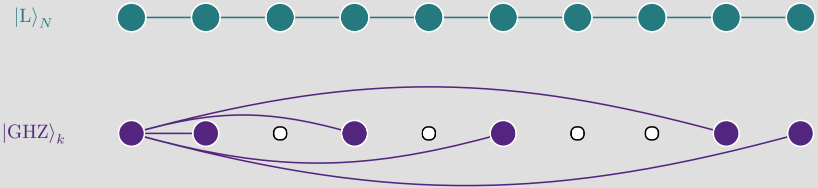


Upper bound



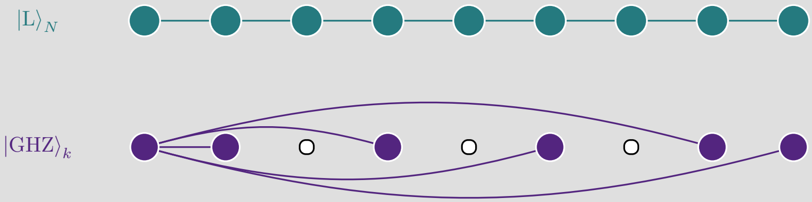
- $k \leq \frac{n+3}{2}$ (odd n)

Upper bound



- $k \leq \frac{n+3}{2}$ (odd n)
- $k \leq \frac{n+2}{2}$ (even n)

Upper bound



- $k \leq \frac{n+3}{2}$ (odd n)
- $k \leq \frac{n+2}{2}$ (even n)

$\rightarrow$ $k \leq \lfloor \frac{n+2}{3} \rfloor$

Maximal pattern

- Measure every node in σ_x basis

[1]See https://github.com/hahnfrederik/Extracting-maximal-entanglement-from-linear-cluster-states

Maximal pattern

- Measure every node in σ_x basis
- Rotations are fairly straightforward[1]

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Other subsets

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- Only constraint is the 2-island

Other subsets

- Shrink the linear cluster state

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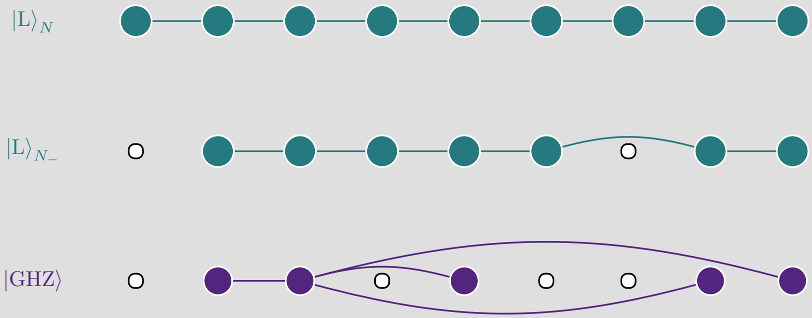
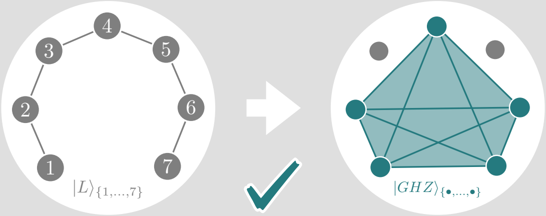
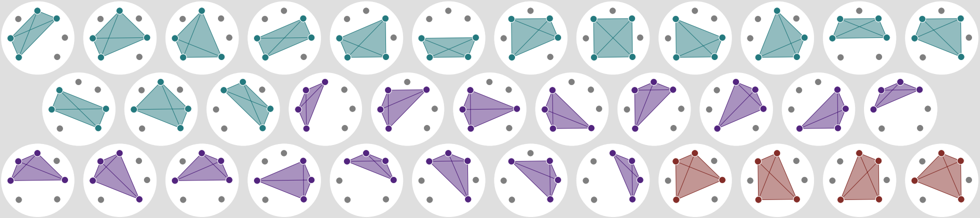


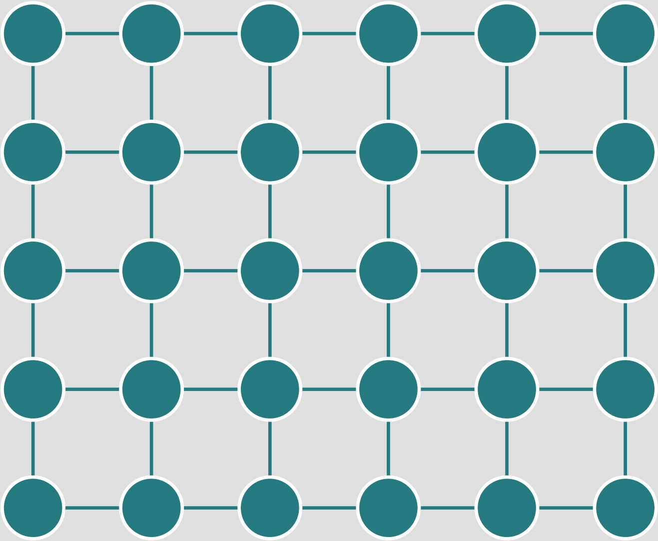
(Small caveat at the edges)

7-PARTITE LINEAR CLUSTER STATE

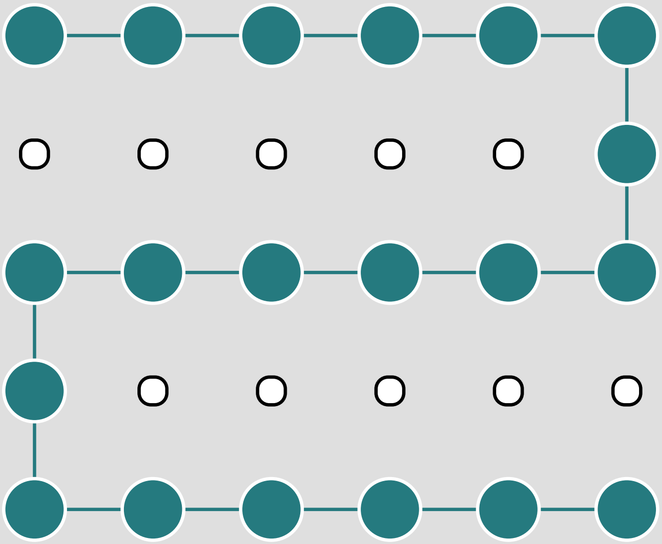




2D-CLUSTER STATE...



2D-CLUSTER STATE...



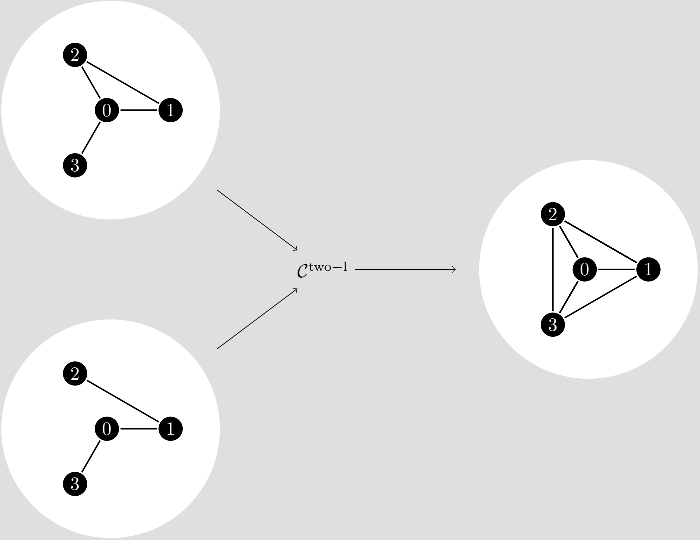
…AND BEYOND

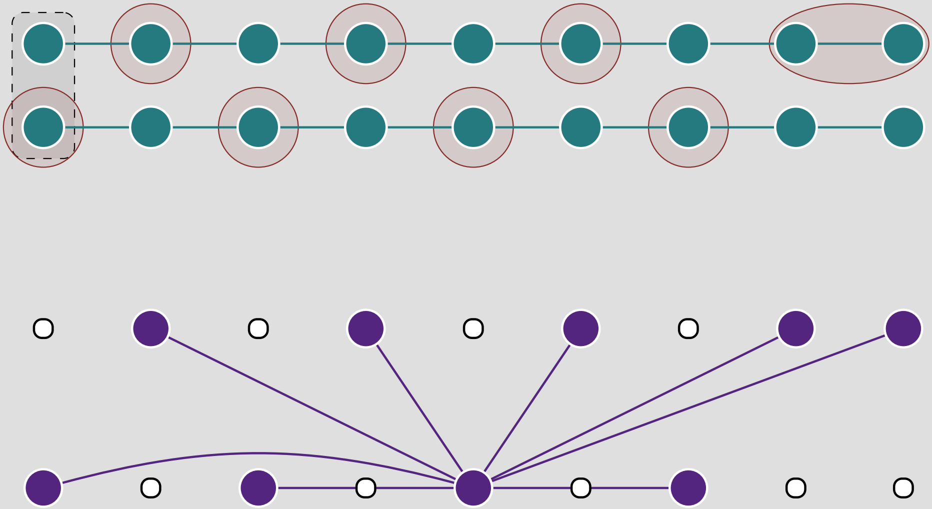
- Add structure
  - Marginals of the states

...AND BEYOND

- Add structure
  - Marginals of the states
- (Less) classical communication
- Two-local Cliffords

Two-local Cliffords





Thank you

quantum
communication
cryptography

berlin

Thank you

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(But if you insist)