Graph states

Graph states as quantum networks

June 14, 2022



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Consider a network



Consider a network



Consider a network



Are these three states 'mappable'?

- |1100 angle + |1101 angle $+ |1110\rangle + |1111\rangle$ (1,2)(2,3)

+ |1000
angle - |1001
angle $+ |1010\rangle + |1011\rangle$

(3,4) (4,1)

- $|0110\rangle + |0111\rangle$
- $+ |0100\rangle + |0101\rangle$
- $+ |0010\rangle |0011\rangle$
- $+ |0000\rangle + |0001\rangle$

- $+ |1100\rangle |1101\rangle$ $- |1110\rangle - |1111\rangle$
- |1010
 angle + |1011
 angle
- $+ |1000\rangle + |1001\rangle$
- $|0110\rangle |0111\rangle$
- $+ |0100\rangle |0101\rangle$
- $+ |0010\rangle |0011\rangle$
- $+ |0000\rangle + |0001\rangle$
- $+ |0000\rangle + |0001\rangle$

 $+ |1110\rangle - |1111\rangle$

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- -|1100
 angle + |1101
 angle
- + |1010
 angle |1011
 angle
- $+\left|1000
 ight
 angle-\left|1001
 ight
 angle$
- $|0110\rangle |0111\rangle$
- $+ |0100\rangle + |0101\rangle$
- $+ |0010\rangle + |0011\rangle$

Graph states

- (1,3)(2,3)(3,4)(2,4)
- (1,2)(2,3)3

Graph states

- Stabiliser states Binary representati
- Transforming stabiliser states
- Graph states Graphs and states Transforming grap
- The end?
- So what's next?
- Other slides

Stabiliser states

- Binary representation
- 2 Transforming stabiliser states

Graph states

- Graphs and states
- Transforming graphs

The end?

5 So what's next?

6 Other slides

Table of Contents

Graph states

Stabiliser states

- Binary representatior
- Transformin stabiliser states
- Graph states Graphs and states Transforming grapt
- The end?
- So what's next?
- Other slides

Stabiliser states

- Binary representation
- Transforming stabiliser states

Graph states

- Graphs and states
- Transforming graphs

The end?

5 So what's next?

6 Other slides

Preliminaries

Graph states

Stabiliser states

Binary representation

Transforming stabiliser states

Graph states Graphs and states Transforming graph

The end?

So what's next?

Other slides

Pauli group

$$\mathcal{P}_n = \{1, -1, +i, -i\} \otimes \{I, X, Y, Z\}^{\otimes n}$$

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Typically we 'forget' about the phases

Consider some states

Graph states

$$|0+1
angle \qquad |0000
angle + |1111
angle \qquad |0000
angle - |1111
angle$$

Stabiliser states

Binary representation

Transformin stabiliser states

Graph states Graphs and states Transforming grap

The end?

So what's next?

Other slides

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Stabiliser states

Graph states

Shared +1 eigenspace

Stabiliser states

The set $\mathcal{S} \subset \mathcal{P}_n$ of a *stabiliser* state $|\psi\rangle$:

$$\mathcal{S} = \{ P \in \mathcal{P}_n | P | \psi \rangle = (+1) | \psi \rangle \}$$

|0000 angle+|1111 angle

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The stabiliser

 \mathcal{S} is an Abelian subgroup of \mathcal{P}_n of order 2^n

Inclusion in stabiliser

Encode Paulis into bitvectors

Graph states

Stabiliser states Binary representation

Transformin stabiliser states

Graph states Graphs and states Transforming graph

The end?

So what's next?

Other slides

Encoding of Paulis



Some nice properties



Stabiliser states Binary representation

Transforming stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Multiplication

$$P_1P_2 \leftrightarrow \overrightarrow{p_1} \oplus \overrightarrow{p_2}$$

$$[P_1, P_2] = 0 \leftrightarrow \overrightarrow{p_1}^T \mathbf{P} \overrightarrow{p_2} = \langle p_1, p_2
angle_s$$

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Generator matrix

Graph states

Stabiliser states Binary representation

Transforming stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Encoding of a set of generators

$$P_1, P_2 \dots P_n\} \to \{\overrightarrow{g_1}, \overrightarrow{g_2} \dots \overrightarrow{g_n}\} \to G = \begin{bmatrix} \mathcal{Z} \\ \mathcal{X} \end{bmatrix} \in \mathbb{F}_2^{2n \times n}$$

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Example of generator matrix

Graph states
States Binary representation

Properties of the generator matrix



Table of Contents

Graph states

Stabiliser states Binary representat

Transforming stabiliser states

- Graph states Graphs and states Transforming graph
- The end?
- So what's next?
- Other slides

Stabiliser states

• Binary representation

2 Transforming stabiliser states

Graph states

- Graphs and states
- Transforming graphs

The end?

5 So what's next?

6 Other slides

Clifford operations

Graph states

Stabiliser states Binary representatio

Transforming stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Operations on stabiliser states

 $\ket{\psi}
ightarrow U \ket{\psi}$

Clifford operators

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Local Cliffords

Graph states

Stabiliser states Binary representatio

Transforming stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Single-qubit operations

LC-equivalent

$$\ket{\psi} = U \ket{\phi}$$
 for some $U \in C_n^L$

Generators

Clifford operations in the binary picture

Graph states	
	Linear mapping
Stabiliser states Binary representation	
Transforming stabiliser states	
Graph states Graphs and states Transforming graphs	Properties of Q
The end?	
So what's next?	
Other slides	

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Binary representation of local Cliffords



Block structure

Q

Transforming stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

$$= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{bmatrix}$$
, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ all diagonal

Equivalence of stabiliser states

 G_1 and G_2 locally equivalent?

Table of Contents

Graph states

- Stabiliser states Binary representat
- Transforming stabiliser states
- Graph states
- Graphs and states Transforming graph
- The end?
- So what's next?
- Other slides

Stabiliser states

- Binary representation
- Transforming stabiliser states

③ Graph states

- Graphs and states
- Transforming graphs

The end?

5 So what's next?

6 Other slides

Graphs

Graph states

Stabiliser states Binary representation

Transforming stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Simple graphs

A graph G = (V, E) is a collection of nodes V and edges E between them.

Neighbourhood

The *neighbourhood* of a node *i*: $N_i = \{a \in V | (a, i) \in E\}$

Adjecency matrix

$$\Theta \in \mathbb{F}_2^{|m{V}| imes |m{V}|}: \Theta_{ij} = 1 \leftrightarrow (i,j) \in E$$

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Graph states



Stabiliser states Binary representatio

Transformin; stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Nodes

Each node represents a qubit in $\left|+\right\rangle$ state

Edges

Each edge (i, j) represents a control- $Z_{(i,j)}$

The graph state		

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Graph states are stabiliser states

Graph states

Stabiliser states Binary representatio

Transforming stabiliser states

Graph states Graphs and states Transforming graph

The end?

So what's next?

Other slides

Generators

$$\{G_i\}=\{X^iZ^{N_i}\}_i$$

Generator matrix



Stabiliser and graph states

Every stabiliser state is **LC-equivalent** to 'a' graph state

Local complementations

Graph states
Graphs and states Transforming graphs

Orbit

Graph states
Transforming graphs

And the associated graph states?

Graph states

Stabiliser states Binary representatio

Transformin stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

The action on the graph...

 $G \xrightarrow{\tau_i} G'$

...has an equivalent action on the graph state

$$\ket{G} \xrightarrow{U_{ au_i} \in C'_n} \ket{G'}$$

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LC-equivalence of graph states

Graph states Statement on *graphs* G_1 and G_2 in each others orbit Transforming graphs Statement on graph states

Other slides

 $|G_1
angle$ and $|G_2
angle$ are LC-equivalent

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Are these three states <u>'mappable'</u> LC-equivalent?

(1,2)(2,3)(1,3)(2,3)(1,2)(2,3)(3,4) (4,1) (3,4)(2,4)- 美 の へ (や 27/39

- $+ |1010\rangle + |1011\rangle$ Transforming graphs - |1100
 angle + |1101
 angle $+ |1110\rangle + |1111\rangle$

Graph states

- $+ |0000\rangle + |0001\rangle$ $+ |0010\rangle - |0011\rangle$ $+ |0100\rangle + |0101\rangle$

 $- |0110\rangle + |0111\rangle$

+ |1000
angle - |1001
angle

- $|1110\rangle |1111\rangle$
 - $+ |1100\rangle |1101\rangle$
- |1010
 angle + |1011
 angle
- $+ |1000\rangle + |1001\rangle$
- $|0110\rangle |0111\rangle$
- $+ |0100\rangle |0101\rangle$
- $+ |0010\rangle |0011\rangle$

- $+ |0000\rangle + |0001\rangle$
- $+ |0000\rangle + |0001\rangle$

- -|1100
 angle + |1101
 angle $+ |1110\rangle - |1111\rangle$
- + |1010
 angle |1011
 angle
- + |1000
 angle |1001
 angle
- $|0110\rangle |0111\rangle$
- $+ |0100\rangle + |0101\rangle$
- $+ |0010\rangle + |0011\rangle$

Are these three states 'mappable' LC-equivalent?

Graph states

Stabiliser states Binary representa

Transformir stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

(1,2) $(2,3)$	(1,3) $(2,3)$	(1,2) (2,
(3,4) $(4,1)$	(3,4) $(2,4)$	(1, 4)

3)

Checking orbits

Transforming graphs

Graph states
Stabiliser
states
And the more general case?
A Bouchet Ap officient algorithm to recognize

A. Bouchet, An efficient algorithm to recognize locally equivalent graphs, Combinatorics 11 (1991)

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Table of Contents

Graph states

- Stabiliser states Binary representat
- Transforming stabiliser states
- Graph states Graphs and states Transforming graph

The end?

So what's next?

Other slides

Stabiliser states

- Binary representation
- Transforming stabiliser states

Graph states

- Graphs and states
- Transforming graphs

The end?

5 So what's next?

Other slides

The end?



Stabiliser states Binary representat

Transformir stabiliser states

Graph states Graphs and states Transforming graph

The end?

So what's next?



Table of Contents

Graph states

- Stabiliser states Binary representat
- Transforming stabiliser states
- Graph states Graphs and states Transforming grap
- The end?
- So what's next?
- Other slides

Stabiliser states

- Binary representation
- Transforming stabiliser states

Graph states

- Graphs and states
- Transforming graphs

The end

5 So what's next?

6 Other slides

Semi-local graph state equivalence

Stabiliser states Binary representat Transforming stabiliser states Graph states Graph states Transforming grap The end? So what's next?
Stabiliser states Binary representat Transforming stabiliser states Graph states Graph states Transforming grap The end? So what's next?
Stabiliser states Binary representat Transforming stabiliser states Graph states Graph states Transforming grap The end? So what's next?
Transforming stabiliser states Graph states Graphs and states Transforming grap The end? So what's next?
Graph states Graphs and states Transforming grap The end? So what's next?
The end? So what's next?
So what's next?

Other tools



Stabiliser states Binary representatio

Transforming stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Measurements on graph states

(Pauli) measurement fit nicely

The graph state basis

A complete basis of different graph states

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Stabiliser codes

Non-complete generator sets

Table of Contents

Graph states

- Stabiliser states Binary representat
- Transforming stabiliser states
- Graph states Graphs and states Transforming grap
- The end?
- So what's next?
- Other slides

Stabiliser states

- Binary representation
- Transforming stabiliser states

Graph states

- Graphs and states
- Transforming graphs

The end

5 So what's next?



Why LC equivalence?

Graph states The LU-LC conjecture LU - LC equivalence SLOCC SLOCC is equivalent to LU

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Other slides

Each stabiliser state is LC-equivalent to a graph state

Graph states

Stabiliser states Binary representati

Transformir stabiliser states

Graph states Graphs and states Transforming grap

The end?

So what's next?

Other slides

$$G = \begin{bmatrix} Z \\ X \end{bmatrix} \xrightarrow{L.C.} G = \begin{bmatrix} Z' \\ X' \end{bmatrix} \xrightarrow{C.O.B.} \begin{bmatrix} Z'X'^{-1} \\ I \end{bmatrix}$$

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Writing down the stabiliser state

Graph states

Stabiliser states Binary representation

Transformin stabiliser states

Graph states Graphs and states Transforming graphs

The end?

So what's next?

Other slides

Either the ket state

$$\ket{\psi} = \prod_{g_i} rac{I+g_i}{2} \left| \overrightarrow{0}
ight
angle$$

Or the density matrix

$$ho = \prod_{g_i} rac{l+g_i}{2} = rac{1}{2^n} \sum_{S \in \mathcal{S}} S$$

Degenerate generator matrices

Graph states

Other slides

Equivalent statements

- The generator matrix G is not full rank
- The associated eigenspace is of higher dimension
- The 'generators' are linearily dependent

A stabiliser code

Generator matrix with rank k < n specifies a 2^k-dimensional subspace

Errors

Any operator that doesn't commute with at least one generator maps **the entire subspace** to an orthogonal space.