Graph states as quantum networks

June 14, 2022

quantum communication cryptography

Consider a network

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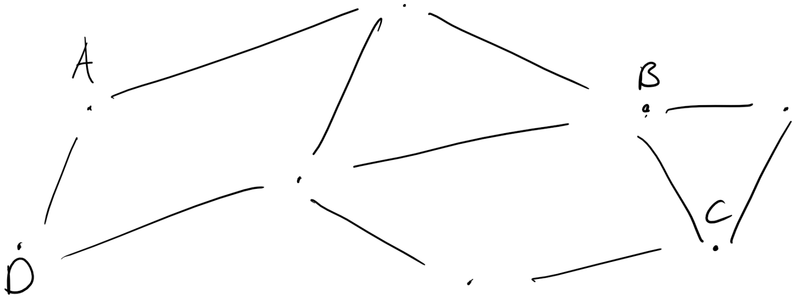

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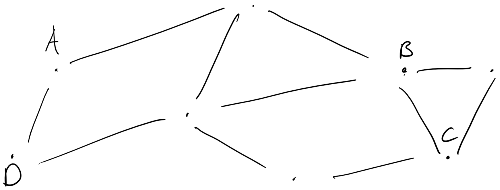

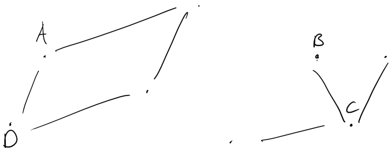

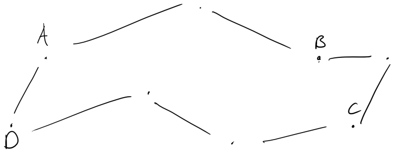

Are these three states 'mappable'?

$$\begin{aligned}&+|0000\rangle+|0001\rangle\\&+|0010\rangle-|0011\rangle\\&+|0100\rangle+|0101\rangle\\&-|0110\rangle+|0111\rangle\\&+|1000\rangle-|1001\rangle\\&+|1010\rangle+|1011\rangle\\&-|1100\rangle+|1101\rangle\\&+|1110\rangle+|1111\rangle\end{aligned}$$

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$(1,2)\ (2,3)$
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$(1,4)$

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Preliminaries

Pauli group

$$\mathcal{P}_n = \{1, -1, +i, -i\} \otimes \{I, X, Y, Z\}^{\otimes n}$$

Typically we 'forget' about the phases

Consider some states

$$|0+1\rangle \qquad |0000\rangle + |1111\rangle \qquad |0000\rangle - |1111\rangle$$

Stabiliser states

Shared $+1$ eigenspace

The set $\mathcal{S} \subset \mathcal{P}_n$ of a *stabiliser* state $|\psi\rangle$:

$$\mathcal{S} = \{P \in \mathcal{P}_n | P\,|\psi\rangle = (+1)\,|\psi\rangle\}$$

The stabiliser

$\mathcal{S}$ is an Abelian subgroup of $\mathcal{P}_n$ of order 2^n

Inclusion in stabiliser

$$|0000\rangle + |1111\rangle$$

Encode Paulis into bitvectors

Encoding of Paulis

$$I \to \begin{bmatrix} 0 \\ \hline 0 \end{bmatrix}$$

$$X \to \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix}$$

$$Y \to \begin{bmatrix} 1 \\ \hline 1 \end{bmatrix}$$

$$Z \to \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix}$$

Some nice properties

Multiplication

$$P_1 P_2 \leftrightarrow \overrightarrow{p_1} \oplus \overrightarrow{p_2}$$

Commutation

$$[P_1, P_2] = 0 \leftrightarrow \overrightarrow{p_1}^T \mathbf{P} \overrightarrow{p_2} = \langle p_1, p_2 \rangle_s$$

Generator matrix

Encoding of a set of generators

$$\{P_1, P_2 \ldots P_n\} \to \{\vec{g_1}, \vec{g_2} \ldots \vec{g_n}\} \to G = \left[\begin{array}{c} \mathcal{Z} \\ \hline \mathcal{X} \end{array}\right] \in \mathbb{F}_2^{2n\times n}$$

Example of generator matrix

Properties of the generator matrix

$P \stackrel{?}{\in} \mathcal{S}$

Completeness of stabiliser?

Change of generators

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Clifford operations

Operations on stabiliser states

$$|\psi\rangle \to U\,|\psi\rangle$$

Clifford operators

Local Cliffords

Single-qubit operations

LC-equivalent

$|\psi\rangle = U|\phi\rangle$ for some $U \in C_n^L$

Generators

Clifford operations in the binary picture

Linear mapping

Properties of Q

Binary representation of local Cliffords

Block structure

$$Q = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right], \ \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \text{ all diagonal}$$

Equivalence of stabiliser states

G_1 and G_2 locally equivalent?

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Graphs

Simple graphs

A *graph* $G = (V, E)$ is a collection of nodes V and edges E between them.

Neighbourhood

The *neighbourhood* of a node i:
$N_i = \{a \in V | (a, i) \in E\}$

Adjecency matrix

$\Theta \in \mathbb{F}_2^{|V| \times |V|} : \Theta_{ij} = 1 \leftrightarrow (i, j) \in E$

Graph states

Nodes

Each node represents a qubit in $|+\rangle$ state

Edges

Each edge (i,j) represents a control-$Z_{(i,j)}$

The graph state

Graph states are stabiliser states

Generators

$$\{G_i\} = \{X^i Z^{N_i}\}_i$$

Generator matrix

$$G = \left[\begin{array}{c} \Theta \\ \hline I \end{array}\right]$$

Stabiliser and graph states

Every stabiliser state is **LC-equivalent** to 'a' graph state

Local complementations

Orbit

And the associated graph states?

The action on the graph...

$$G \xrightarrow{\tau_i} G'$$

...has an equivalent action on the graph state

$$|G\rangle \xrightarrow{U_{\tau_i} \in C_n^l} |G'\rangle$$

LC-equivalence of graph states

Statement on *graphs*

G_1 and G_2 in each others orbit

Statement on *graph states*

$|G_1\rangle$ and $|G_2\rangle$ are LC-equivalent

Are these three states ~~‘mappable’~~ LC-equivalent?

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Checking orbits

And the more general case?

A. Bouchet, An efficient algorithm to recognize locally equivalent graphs, Combinatorics 11 (1991)

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The end?

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Semi-local graph state equivalence

Other tools

Measurements on graph states

(Pauli) measurement fit nicely

The *graph state basis*

A complete basis of different graph states

Stabiliser codes

Non-complete generator sets

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Why LC equivalence?

The LU-LC conjecture

LU - LC equivalence

SLOCC

SLOCC is equivalent to LU

Each stabiliser state is LC-equivalent to a graph state

$$G = \begin{bmatrix} Z \\ \hline X \end{bmatrix} \xrightarrow{L.C.} G = \begin{bmatrix} Z' \\ \hline X' \end{bmatrix} \xrightarrow{C.O.B.} \begin{bmatrix} Z'X'^{-1} \\ \hline I \end{bmatrix}$$

Writing down the stabiliser state

Either the ket state

$$|\psi\rangle = \prod_{g_i} \frac{I + g_i}{2} \left|\vec{0}\right\rangle$$

Or the density matrix

$$\rho = \prod_{g_i} \frac{I + g_i}{2} = \frac{1}{2^n} \sum_{S \in \mathcal{S}} S$$

Degenerate generator matrices

Equivalent statements

- The generator matrix G is not full rank
- The associated eigenspace is of higher dimension
- The 'generators' are linearily dependent

A stabiliser *code*

Generator matrix with rank $k < n$ specifies a 2^k-dimensional subspace

Errors

Any operator that doesn't commute with at least one generator maps **the entire subspace** to an orthogonal space.