Classification of graph state orbits by their marginal structure

1. Marginal states

Let $M \subset V$ be any subset of the nodes of V and $M^\perp = V \setminus M$. The *marginal* on M is the reduced state

$$\rho_M = \mathrm{tr}_{M^\perp}\left[\, |G\rangle\langle G| \,\right]$$

The *rank* of a marginal is invariant under local unitary operations. If $|G_1\rangle \overset{\text{L.U.}}{=} |G_2\rangle$, then

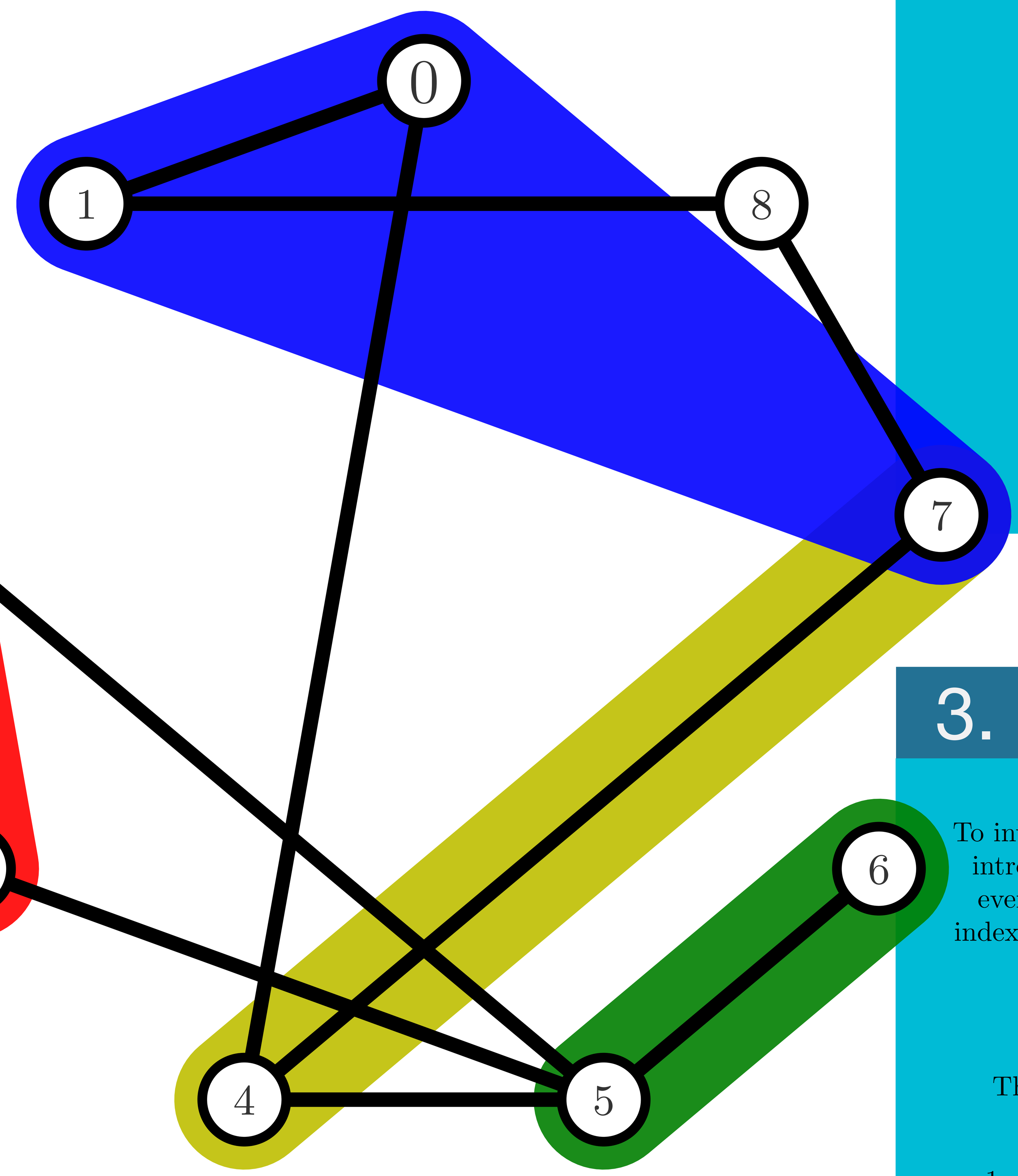
$$\mathrm{rank}\,(\rho_{1,M}) = \mathrm{rank}\,(\rho_{2,M})$$

We call a marginal **non-trivial** if $\mathrm{rank}\,(\rho_M) < 2^{|M|}$.

Non-trivial Paulis are traceless; we collect the elements without support only in M to get a set $\mathcal{S}_M$. Then

$$\rho_M \propto \mathrm{tr}_{M^\perp}\Big[\sum_{\sigma\in\mathcal{S}} \sigma\Big] \propto \sum_{\mathcal{S}_M} \sigma$$

$\mathcal{S}_M$ is an Abelian subgroup of $\mathcal{S}$and thus forms a stabilizer code; ρ_M is exactly the maximally mixed state in its codespace.



4. Classification

We now study how the signatures perform in discerning the graph state entanglement classes of size n. For a given k, we calculate the ratio of unique signatures, i.e. the total number of unique signatures divided by the total number of unique classes for a given n.

We find the following ratios:

n	t_2	t_3	t_4	t_5	t_6
5	1.00	1.00	1.00	–	–
6	0.73	1.00	1.00	1.00	–
7	0.46	1.00	1.00	1.00	1.00
8	0.19	0.89	1.00	1.00	1.00
9	0.06	0.73	0.998	0.998	0.998
10	0.01	0.37	0.988	0.999	0.999

2. Using the graph

There thus is a $1:1$ correspondence:

$$\mathrm{rank}\,(\rho_M) = 2^{|M|} - |\mathcal{S}_M| = 2^{|M|-n_M},$$

where $n_M := \log(\mathcal{S}_M)$; it can be computed as the nullity of a submatrix of Γ, the adjacency matrix.

Any $\sigma \in \mathcal{S}_M$ **uniquely** corresponds to a **subset** of M that has an **even number** of edges to *all* nodes in $M^\perp$. This allows us to calculate $|\mathcal{S}_M|$ by looking at a graph and checking all subsets of M.

For the highlighted marginals:

M	Surving subsets	$\lvert\mathcal{S}_M\rvert$	$\mathrm{rank}\,(\rho_M)$
$\{2,3\}$	$\{\},\{2,3\}$	2	2
$\{5,6\}$	$\{\},\{6\}$	2	2
$\{0,1,7\}$	$\{\},\{0,1,7\}$	2	4
$\{4,7\}$	$\{\}$	1	4

3. Smart bookkeeping

To inventarize all ranks of marginals of a fixed size k, we introduce a k-dimensional tensor T_k^G with length n in every dimension for every graph state $|G\rangle$. For every index-as-a-vector $\mathbf{u}$ of T_k, let $M(\mathbf{u}) \subset V$ be the set of its unique elements. T_k^G is defined by

$$T_k^G(\mathbf{u}) = \mathrm{rank}\,\big(\rho_{M(\mathbf{u})}\big)$$

This tensor is constant for $L.U.$-orbits, but not for permutations; we derive a *signature*:

1. Compute the matrix $K = \underbrace{\mathrm{sum}(\ldots(\mathrm{sum}}_{k-2 \text{ times}}(T_k)\ldots)^*$
2. Compute the eigenvalues λ_i of K; these are permutation invariant. Discard any $\lambda_i = 0$
3. Define $t_k = \prod \lambda_i$: the (k-th) *signature* of the entanglement class

*Here, *sum* means a contraction over a dimension.

There are two 3-body marginals with rank 2

Can you find them both?

Theory, background info & introductions

Graph states

A graph is a collection of *nodes* V and *edges* $E \subseteq V^2$ between them; we say $n := |V|$.

We now associate a qubit in the $|+\rangle$ state with every node of G. The *graphstate* $|G\rangle$ is the unique eigenstate of the n operators

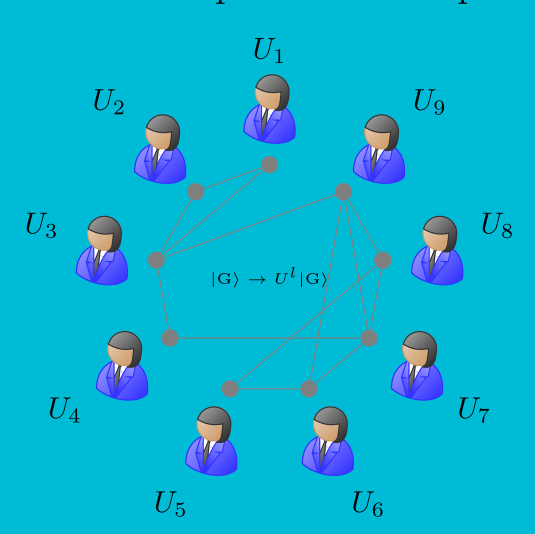
$$\{g_i = X_i \bigotimes_{j\in N_i} Z_j\} \quad \forall i \in V,$$

i.e. an X operator on qubit i and a Z operator on every node connected to i. These n operators form a generating set for a *stabilizer* $\mathcal{S}$, and thus any graph state is also a stabiliser state; therefore

$$|G\rangle\langle G| \propto \sum_{\mathcal{S}} \sigma$$

Local operations

Local unitaries are tensorproducts of 1-qubit unitaries U_i:



If two graph states are $L.U.$ equivalent, we write $|G_1\rangle \overset{L.U.}{=} |G_2\rangle$.

Entanglement classes

For a given graphstate $|G\rangle$, its **(L.U.)-orbit** is the collection of all states LU-equivalent to $|G\rangle$.

In the **classification of entanglement**, states that are in each others orbit or equal up to a permutation of the qubits are grouped together in disjoint **entanglement classes**.

For 2-qubit states there are two: separable, and the Bell state. For 3-qubit states there are two more: the GHZ- and W states. In general, there are exponentially many.

We focus on entanglement classes containing graph states, like the GHZ state.

Universität Siegen, Technische Universität Berlin, Technische Universität Berlin, CWI, Universität Siegen, Technische Universität Berlin

L. Vandré, **J. de Jong**, F. Hahn, A. Burchardt, O. Gühne, A. Pappa